Homework 6

1. Thermonuclear Reaction Rates

   We have generalized our discussion somewhat, however the reaction rate is still defined in terms of mono-energetic particles (i.e. the target and projectile particles have fixed velocities, $\vec{v}_t$ and $\vec{v}_p$ respectively). In general, an experimental setup will create beams and targets with a distribution of particle velocities. One then speaks in terms of averaged quantities. The same is true in astrophysical environments such as stars. Usually these environments are in thermal equilibrium, which makes calculating the averages more simple. This brings us to the thermally-averaged reaction rate.

   $$< reaction\ rate > = n_p n_t < \sigma v > \tag{1}$$

   The averaged reaction rate is given by:

   $$< reaction\ rate > = \int \int d^3\vec{v}_p\, d^3\vec{v}_t\, f_p(\vec{v}_p)\, f_t(\vec{v}_t)\, \sigma(E) v \tag{2}$$

   remembering that $n_i = \int d^3\vec{v}_i\, f_i(\vec{v}_i)$, and $f_i(\vec{v}_i) = g_i \left(\frac{m_i}{2\pi\hbar}\right)^3 \exp\left(\frac{\mu_c - E_i}{kT}\right)$, where $E_i = m_i c^2 + \frac{1}{2}m_i v_i^2$.

   (a) What is n_i for a Maxwellian distributed particle with mass m_i at a temperature T?

   (b) With the definitions of the center-of-mass total and relative velocities ($\vec{V} = \frac{m_p \vec{v}_p + m_t \vec{v}_t}{m_p + m_t}$ and $\vec{v} = \vec{v}_p - \vec{v}_t$), and assuming that the particle distributions are Maxwellian (with the same temperature, T), transform the reaction rate integral such that it only depends on center-of-mass velocity variables.

   (c) Show that if the cross-section depends only on the magnitude of the relative velocity (thus E) that you can reduce this integral down to a 1-dimensional integral over v.

   (d) Transform this 1-dimensional integral from being over v to being over E.

   (e) Solve for the quantity $< \sigma v > = < reaction\ rate > /n_p n_t$.

   In chemistry, one speaks in terms of a rate of reaction. The reaction rate is equal to the total rate of change of the nuclides involved:

   $$-\frac{1}{\mathcal{N}_A}\frac{dn_A}{dt} = -\frac{1}{\mathcal{N}_B}\frac{dn_B}{dt} = \frac{1}{\mathcal{N}_C}\frac{dn_C}{dt} = \frac{1}{\mathcal{N}_D}\frac{dn_D}{dt} = < reaction\ rate > \tag{3}$$

   where, $\mathcal{N}_i$ are the number of nuclides of type i produced or destroyed in the reaction $A + B \to C + D$.

   Since nuclear reactions conserve the number of nucleons, only shuffling them around into different nuclides, it is useful to rewrite things in terms of a molar fraction. $Y_i \equiv n_i/n_{baryon}$ is the molar fraction, defined as the number of nuclides of type i per baryon, or the ratio of their number densities. Since the total baryon number is conserved in nuclear reactions, $dn_{baryon}/dt = 0$. Therefore, we can rewrite our equations:

   $$-\frac{1}{\mathcal{N}_A}\frac{dY_A}{dt} = -\frac{1}{\mathcal{N}_B}\frac{dY_B}{dt} = \frac{1}{\mathcal{N}_C}\frac{dY_C}{dt} = \frac{1}{\mathcal{N}_D}\frac{dY_D}{dt} = Y_A Y_B n_{baryon} < \sigma v > \tag{4}$$

   Since the typical mass of a baryon is $\sim m_u$, we can substitute the baryon number density with the baryon mass density ($\rho_{baryon} \approx n_{baryon} m_u$) and Avogadro's number (N_A).

   $$-\frac{1}{\mathcal{N}_A}\frac{dY_A}{dt} = -\frac{1}{\mathcal{N}_B}\frac{dY_B}{dt} = \frac{1}{\mathcal{N}_C}\frac{dY_C}{dt} = \frac{1}{\mathcal{N}_D}\frac{dY_D}{dt} = Y_A Y_B \rho_{baryon} N_A < \sigma v > \tag{5}$$

   When we talk about reaction rates, we usually are referring to the reduced reaction rate $\lambda = N_A < \sigma v >$. This way, we do not have to specify the concentrations of nuclides or the baryon density. We are left with a quantity (λ) that usually only depends on the temperature, T.